

# Scaling and Memory in Return Loss Intervals and Application to Risk Estimation

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## ABSTRACT

We propose simple practical method to estimate the risk level of a certain probability of loss under the definition of Value-at-risk.<sup>1</sup> We study the statistics of the return loss intervals  $\tau_q$  between two consecutive losses below a threshold  $-q$ , in various stocks, currencies and commodities and also some models. We found the statistics of  $\tau_q$  has interesting characteristics, scaling and universality. Scaling allows us to extrapolate rare events from the behavior of more frequent events, with therefore good statistics.<sup>3-6</sup> Universality induces us to think again about 'stylized facts' of financial data.<sup>2</sup> The conditional mean interval which depends on history shows systematic behavior and enables us to estimate the risk level.

**Keywords:** Return loss intervals, Scaling, Universality, Value-at-risk

## 1. INTRODUCTION

The most common indicator of risk in financial world is the Value-at-risk(Var) which is defined by the risk at a level of loss  $\Lambda$

$$\int_{-\infty}^{-\Lambda} p_{\Delta}(r) dr = p^* \quad (1)$$

where  $p_{\Delta}(r)$  is the probability density of return  $r(t) = \log(p(t)/p(t - \Delta))$  on the time interval of  $\Delta$ , and  $p^*$  is the probability of loss. We propose a simple method which enables us to estimate Var using historical information. We study the statistics of the return loss intervals  $\tau_q$  between two consecutive losses below a threshold  $-q$ , in various stocks, currencies and commodities and also some models (for illustration, see Fig.1). Here we apply the scaling properties of the return intervals<sup>3-6</sup> to estimate the risk in extreme events using information from more frequent events.

## 2. SCALING

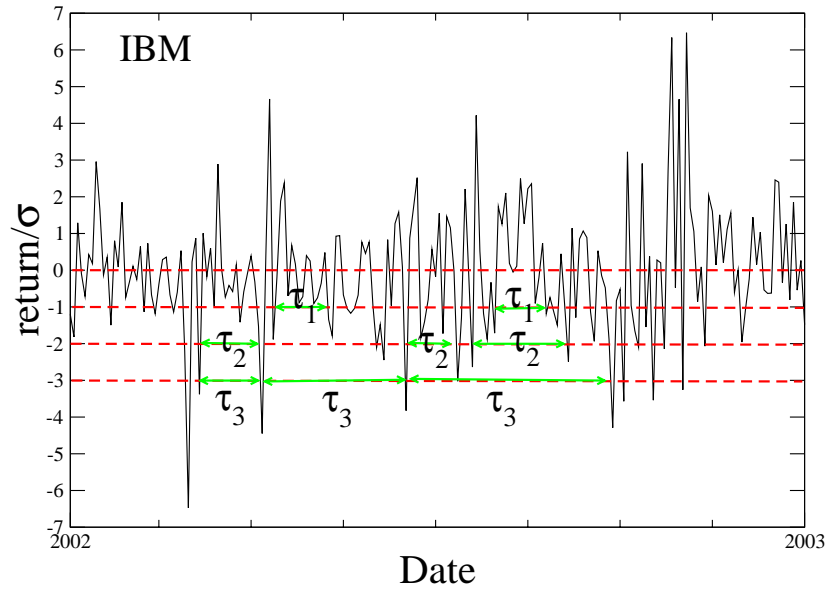
We analyzed representative three daily stocks in NYSE, three daily currencies and three daily commodities. We study the pdf of the return loss intervals,  $P_q(\tau)$ , is not a function of the two independent variables  $\tau$  and  $q$ , but depends only on the scaled parameter  $\tau/\bar{\tau}_q$ , where the  $q$ -dependence is contained in the mean return loss interval

$\bar{\tau}_q \equiv \frac{1}{N_q} \sum_{i=1}^{N_q} \tau_q(i)$  and  $N_q$  is the total number of return loss intervals for a given  $q$ .

The scaling property enables to estimate  $P_q(\tau)$  of large values of  $q$  (rare events) from that of small value of  $q$  (frequent events). These scaling is also found in the volatility of stocks and currencies as well as in various

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**Figure 1.** Schematic illustration of the loss intervals  $\tau_q(l)$ ,  $l = 1, \dots, N_q$  for three threshold values  $q_1 = 1$ ,  $q_2 = 2$  and  $q_3 = 3$  for the return losses of IBM stock prices. The return is normalized with standard deviation  $\sigma$ .

areas ,such earthquake and climate fluctuations which have long term memory with fractality.<sup>3-6</sup>

Fig.2 shows both scaled and unscaled  $P_q(\tau)$  of IBM's stock returns (11700 days). Curves of unscaled  $P_q(\tau)$  collapse to a single curve. Figs. 3,4,5 show the scaled pdf  $P_q(\tau)\bar{\tau}_q$ , of stocks, currencies and commodities as functions of the scaled return loss intervals  $\tau/\bar{\tau}_q$ . Curves of the data from different  $q$  collapse to a single curve, consistent with the scaling relation

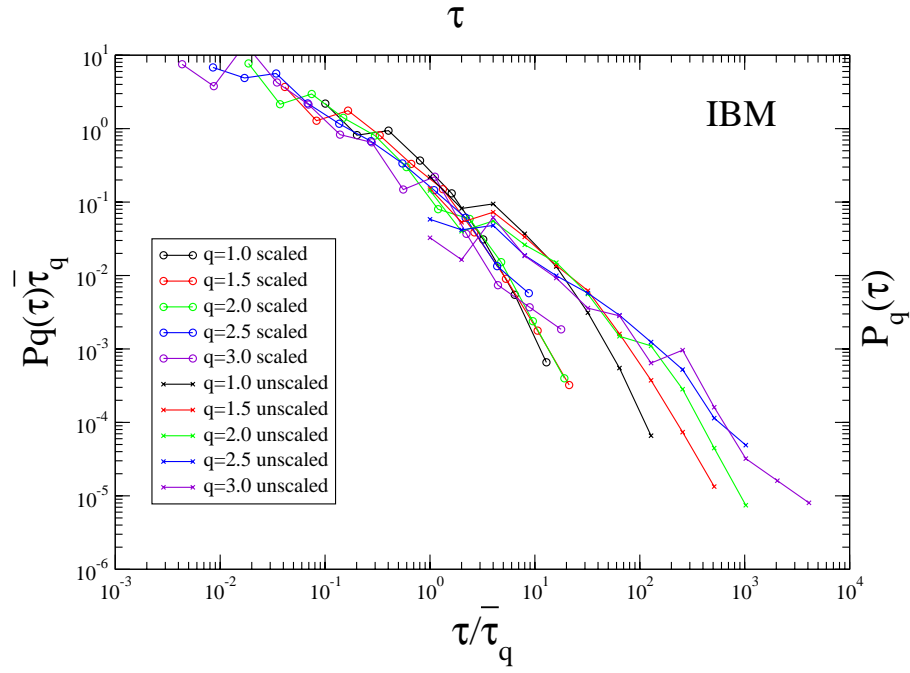
$$P_q(\tau) = \frac{1}{\bar{\tau}_q} f\left(\frac{\tau}{\bar{\tau}_q}\right) \quad (2)$$

The scaling function  $f(x)$  does not depend explicitly on  $q$ . Hence if  $P_q(\tau)$  is known for one value of  $q$ , one can estimate it for other  $q$  — in particular for very large  $q$  (rare events) that are difficult to study due to the lack of data.

### 3. UNIVERSALITY

Our results also show that the scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , is almost same in several stocks, currencies and commodities, suggesting universality in the economic fluctuations. This is different from the return distribution, since shuffled data which is from the same return distribution has different function from the unshuffled one. In Figs. 3,4,5 , curves of the data from different stocks, currencies and commodities respectively collapse to a single function form, while after shuffling the original return records the function is modified. Fig.6 shows the scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of three stocks, three currencies and three commodities with  $q=1.0$  and  $2.0$ . We can see nine curves of unshuffled scaled data clearly collapse to a well approximated single curve.

Fig.7 shows the scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of fractional Brownian motion with Hurst exponent  $h = 0.2 \sim 0.9$  and  $q = 1.0 \sim 3.0$ . As Hurst exponent grows, the curves of the lines become flat. Fig.8 shows the scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of various time series. The shapes are quite different. Actually they have the tendency in which curves become flat as Hurst exponents grow, similar to fractional Brownian motion. But Hurst exponent is not sole element to decide the shape of  $P_q(\tau)\bar{\tau}_q$ . Fig.7 and Fig.8 show this universality is not subject to the return distribution and the Hurst exponent.



**Figure 2.** Both scaled and unscaled  $P_q(\tau)$  of IBM's stock returns are shown. Curves of unscaled  $P_q(\tau)$  collapse to a single curve.

#### 4. ESTIMATION OF VAR

We can easily show relation between  $\bar{\tau}_q$  and Var. Inverse of  $\bar{\tau}_q$  corresponds to

$$\int_{-\infty}^{-\Lambda} p_{\Delta}(r) dr = \frac{\text{number of } \Delta's \text{ with } r < -\Lambda}{\text{total number of } \Delta's} \quad (3)$$

Because

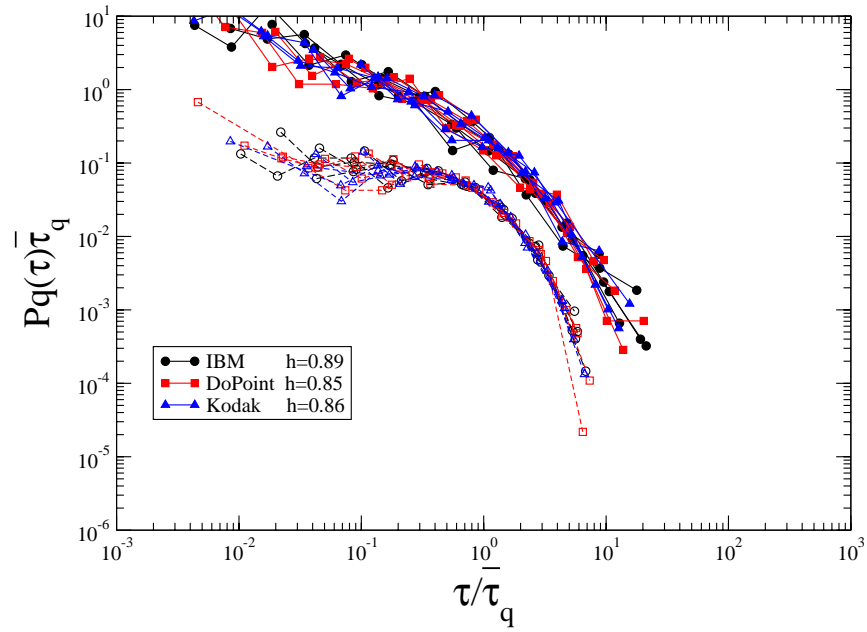
$$\bar{\tau}_q \equiv \frac{1}{N_q} \sum_{i=1}^{N_q} \tau_q(i) \quad (4)$$

And

$$\begin{aligned} \sum_{i=1}^{N_q} \tau_q(i) &\approx \text{total number of } \Delta's \\ N_q + 1 &= \text{number of } \Delta's \text{ with } r < -\Lambda \end{aligned} \quad (5)$$

That means a certain probability of loss  $\bar{\tau}_q^{-1}$  gives the risk level of loss  $-q$ .  $P_q(\tau)$  includes more information than Var. We can also estimate the conditional mean return loss interval  $\bar{\tau}_q (< \tau_{<q} >)$  which enables us dynamic estimation of the risk level of loss  $-q$ . For the conditional mean, we use statistical quantity  $< \tau_{<q} >$  to represent a condition of each moment to get good statistics. Here,  $< \tau_{<q} >$  is average mean interval of  $\bar{\tau}_{q'}, q' = < q$

$$< \tau_{<q} > = \frac{1}{j_q} \sum_{j=1}^{j_q} \bar{\tau}_{q_j} \quad q_j < q, j = 1, \dots, j_q - 1 \quad \text{and} \quad q_{j_q} = q \quad (6)$$



**Figure 3.** The scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of three stocks as functions of the scaled return loss intervals  $\tau/\bar{\tau}_q$  with  $q = 1.0, 1.5, 2.0, 2.5, 3.0$  are shown. Curves of the data from different  $q$  collapse to a single curve. And curves of the data from different stocks also collapse to a single curve. Hurst exponents are shown in the graph.

First we found conditional mean  $\bar{\tau}_q(< \tau_{<q} >)$  is proportional to a certain power of condition  $< \tau_{<q} >$ .

$$\log(\bar{\tau}_q(< \tau_{<q} >)/\bar{\tau}_q) = \phi(q) \log(< \tau_{<q} >) + \varphi(q) \quad (7)$$

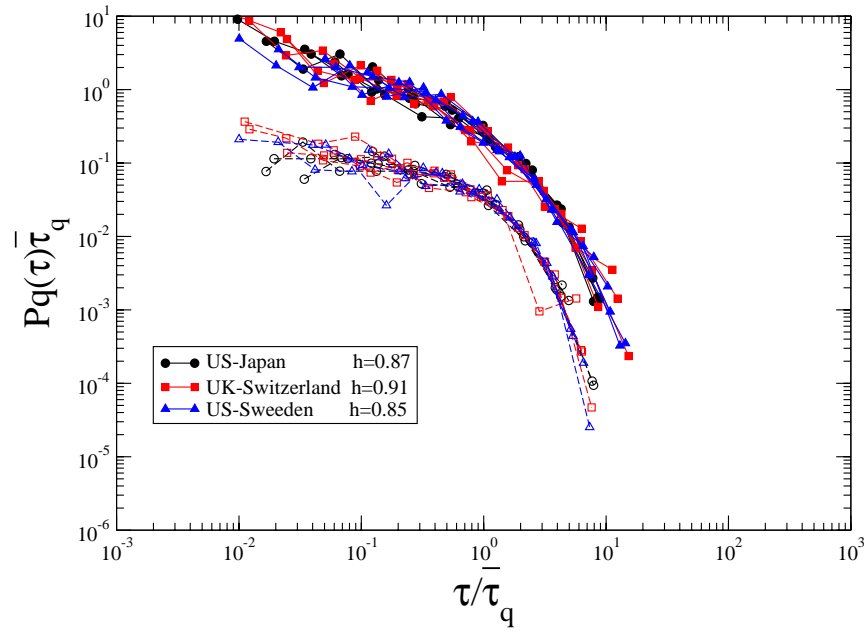
Next we found the power  $\phi(q)$  and coefficient  $\varphi(q)$  have clear and simple dependence of  $q$  as is shown inserted graph of Fig.9.

$$\begin{aligned} \phi(q) &= -0.88 + 1.4q - 0.21q^2 \\ \varphi(q) &= 0.27 - 0.29q + 0.1q^2 \end{aligned} \quad (8)$$

It is well known that cumulative return distribution has power law of -3.

$$\bar{\tau}_q^{-1} = \frac{1}{7.6}q^{-3} \quad (9)$$

The equations (7),(8) and (9) enable us to estimate  $\bar{\tau}_q(< \tau_{<q} >)$  with large value of  $q$ . Fig.10 shows this estimation.  $\bar{\tau}_q(< \tau_{<q} >)$  is the inverse of a certain probability of loss  $p^*$ , so if we want to know the risk level corresponding to 1% probability of loss within the time interval of  $\Delta =$  one day, we only look at intersections between the horizontal line of  $\bar{\tau}_q(< \tau_{<q} >) = 100$  and lines with fixed  $q$  in Fig.10. The condition  $< \tau_{<q} >$  which is the average of previous intervals of losses below  $-q'$  with  $q' = < q$ , is changing every day. So the risk level is changing every day and can be estimated in very easy way. Our method provide really simple practical tool for estimation of risk.



**Figure 4.** The scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of three currencies as functions of the scaled return loss intervals  $\tau/\bar{\tau}_q$  with  $q = 1.0, 1.5, 2.0, 2.5, 3.0$  are shown. Curves of the data from different  $q$  collapse to a single curve. And curves of the data from different currencies also collapse to a single curve. Hurst exponents are shown in the graph.

## 5. CONCLUSION

We study the statistics of the return loss intervals  $\tau_q$  between two consecutive losses below a threshold  $-q$ , in various stocks, currencies and commodities and also some models.

We find that the pdf of the return loss intervals,  $P_q(\tau)$ , is not a function of the two independent variables  $\tau$  and  $q$ , but depends only on the scaled parameter  $\tau/\bar{\tau}_q$ . This scaling allows us to extrapolate rare events from the behavior of frequent events, with therefore good statistics.

Another feature of the scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , is universality that shapes are almost same in several stocks, currencies and commodities.

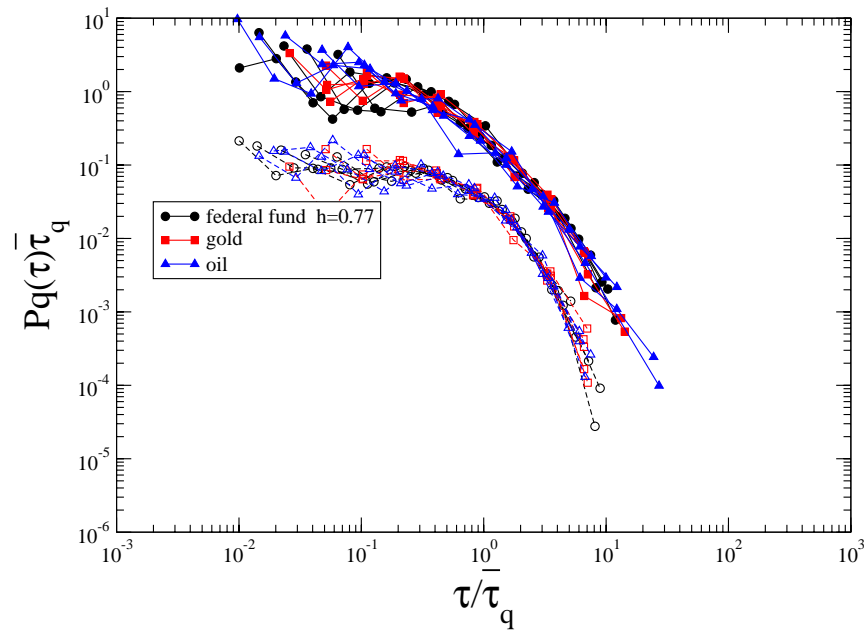
We can easily show relation between  $\bar{\tau}_q$  and Var. We can also estimate the conditional mean return loss interval,  $\bar{\tau}_q(<\tau_{<q}>)$ , which enables us to estimate the risk level of loss  $-q$ . The following equation is supported by the data with good statistics and makes it possible to estimate risk level Var in really simple way through calculated curves in Fig.10.

$$\log(\bar{\tau}_q(<\tau_{<q}>)/\bar{\tau}_q) = \phi(q) \log(<\tau_{<q}>) + \varphi(q) \quad (10)$$

For now we do not know the reason and theory of this scaling and universality and the reason the estimation of the equation by the data shows good statistics. But we can propose simple practical method to estimate the risk level.

## ACKNOWLEDGMENTS

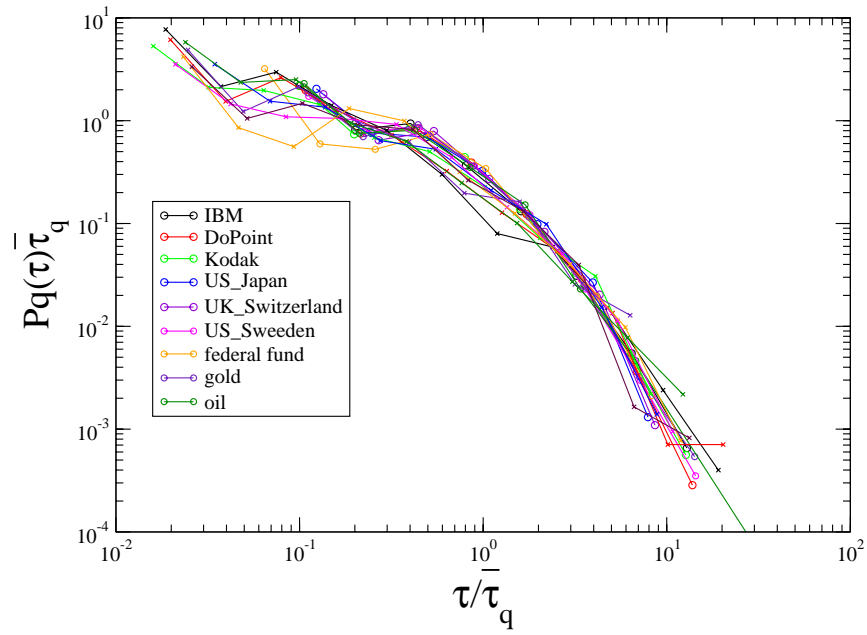
This unnumbered section is used to identify those who have aided the authors in understanding or accomplishing the work presented and to acknowledge sources of funding.



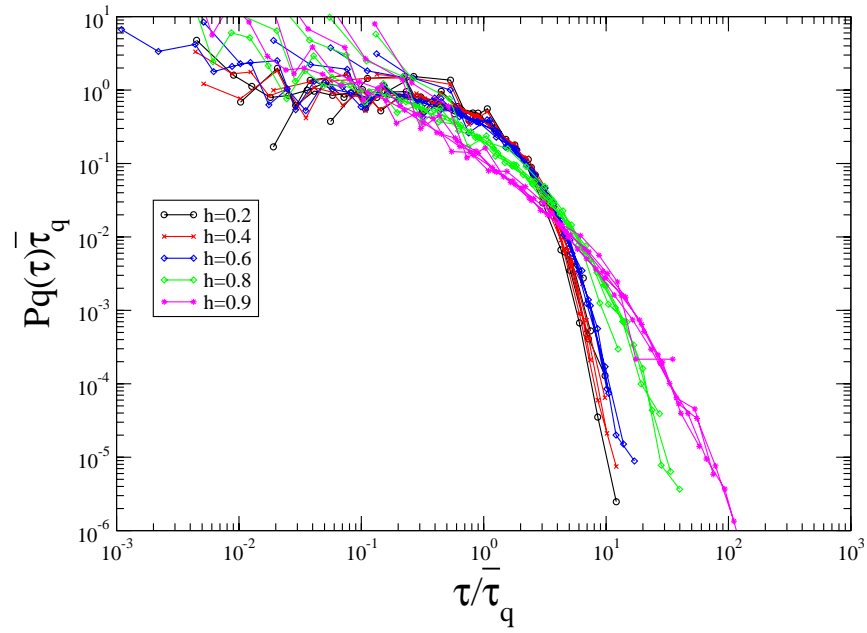
**Figure 5.** The scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of three commodities as functions of the scaled return loss intervals  $\tau/\bar{\tau}_q$  with  $q = 1.0, 1.5, 2.0, 2.5, 3.0$  are shown. Curves of the data from different  $q$  collapse to a single curve. And curves of the data from different commodities also collapse to a single curve. Hurst exponents of federal fund are shown in the graph. The data lengths of gold and oil are short to get correct Hurst exponents.

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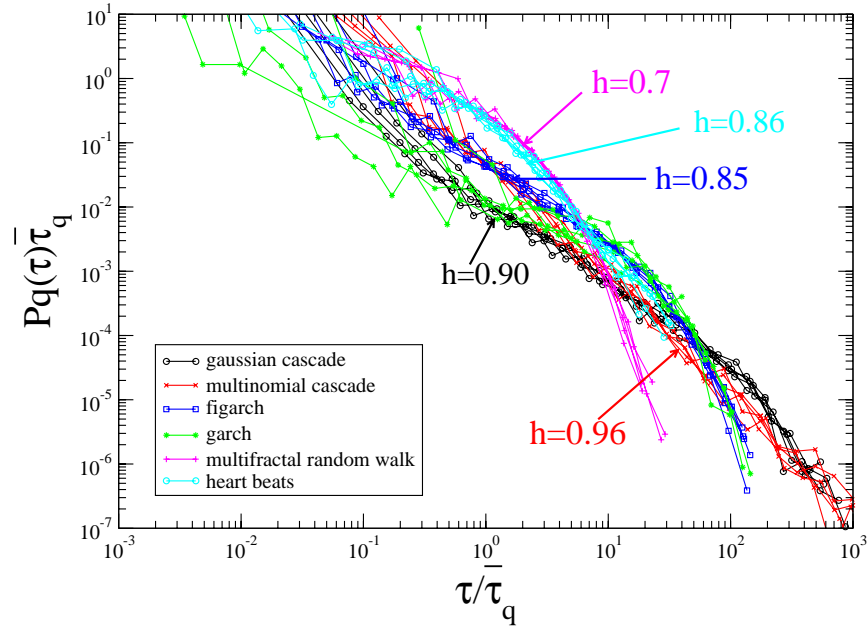
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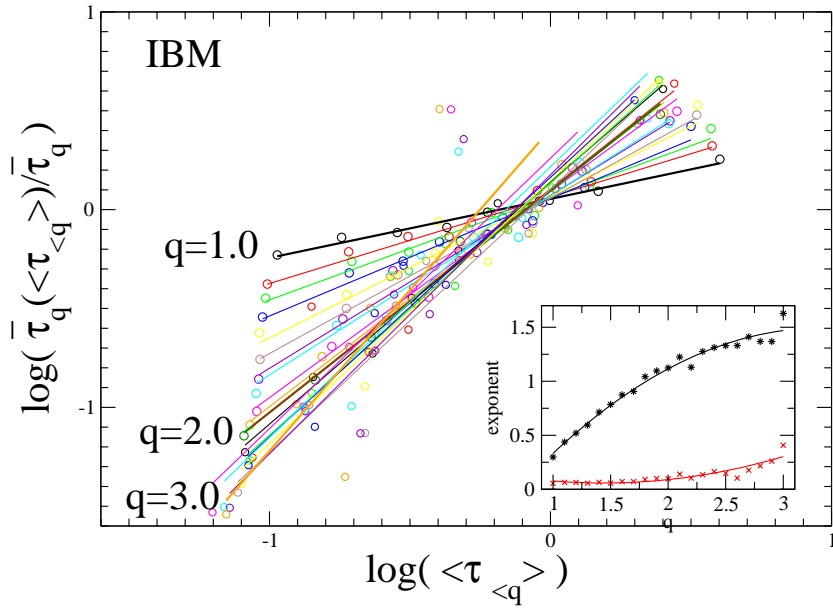
**Figure 6.** Scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of 3 stocks, 3 currencies and 3 commodities with  $q=1.0$  and that of shuffled data are shown. We can see 9 curves with unshuffled data clearly collapse to a curve and shuffling changes their shape.



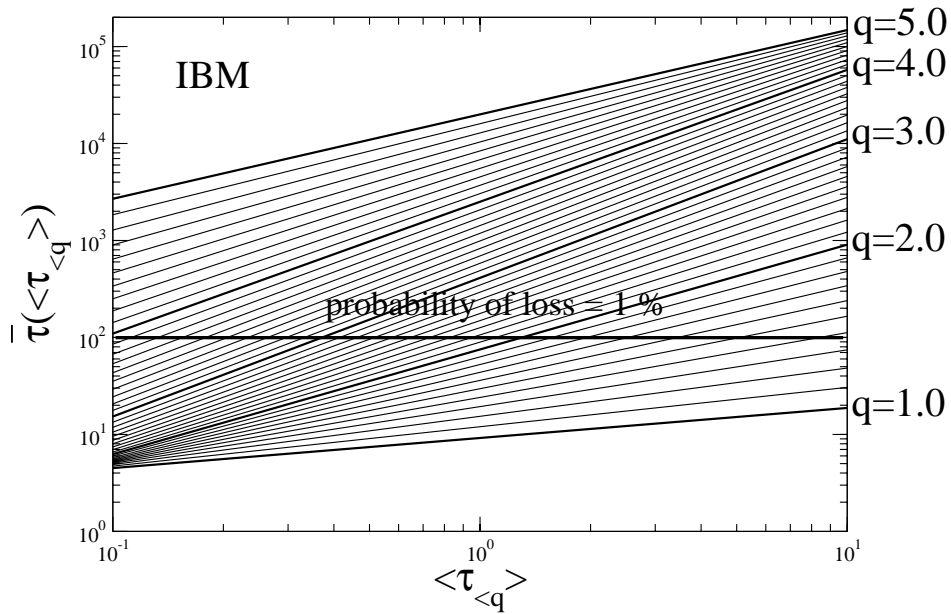
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**Figure 8.** The scaled pdf,  $P_q(\tau)\bar{\tau}_q$ , of various time series are shown. The shapes are quite different. Actually they have tendency that curves become flat as Hurst exponents grow similar to fractional Brownian motion. But Hurst exponent is not sole element to decide the shape of  $P_q(\tau)\bar{\tau}_q$ .



**Figure 9.** Logarithmic expression of the conditional mean of interval  $\bar{\tau}_q(\langle \tau_{<q} \rangle)$  are shown. The condition  $\langle \tau_{<q} \rangle$  is defined by the equation (6). The mean depends on the condition by a certain power and these coefficients can be estimated clearly as quadratic function of  $q$  as equation (8).



**Figure 10.** The estimation of conditional mean using the equations (8),(9) .  $\overline{\tau}_q(\langle \tau_{<q} \rangle)$  is the inverse of a certain probability of loss  $p^*$  so if we want to know the risk level corresponding to 1% probability of loss within the time interval of  $\Delta$ , we only look at intersections between the horizontal line of  $\overline{\tau}_q(\langle \tau_{<q} \rangle) = 100$  and lines with fixed  $q$ . The condition  $\langle \tau_{<q} \rangle$  which is the average of previous intervals of losses below  $-q'$  with  $q' = < q$ , changes at every moment. So the risk level changes every moment and can be estimated in very easy way.