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# The interacting gaps model: reconciling theoretical and numerical approaches to limit-order models

Lev Muchnik<sup>a</sup>, Frantisek Slanina<sup>b</sup>, Sorin Solomon<sup>a,\*</sup>

<sup>a</sup>*Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel*

<sup>b</sup>*Institute of Physics, Academy of Sciences of the Czech Republic, Na Slovance 2, CZ-18221 Praha, Czech Republic*

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## Abstract

We consider the emergence of power-law tails in the returns distribution of limit-order driven markets. We explain a previously observed clash between the theoretical and numerical studies of such models. We introduce a solvable model that interpolates between the previous studies and agrees with each of them in the relevant limit.

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## 1. Introduction

Nowadays, it appears quite natural to model various social phenomena, including economic activity and stock market in particular, using the general reductionist and atomistic paradigm of physics. Indeed, the elementary entities in play may have whatever nature you can imagine and conceptual difference between a system of large number of H<sub>2</sub>O molecules and system of large biochemical supermacromolecules called humans is only matter of scale. On the contrary, the very physical concept of universality may teach us that it may perhaps be deeply irrelevant whether humans are composed of quarks or any other, may be much more tractable stuff.

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\* Corresponding author.

E-mail addresses: [lev@topspin.co.il](mailto:lev@topspin.co.il) (L. Muchnik), [slanina@fzu.cz](mailto:slanina@fzu.cz) (F. Slanina), [sorin@cc.huji.ac.il](mailto:sorin@cc.huji.ac.il) (S. Solomon).

A good deal of considerations like that was pursued in the later decade or two, leading to unprecedented proliferation of various more or less successful sociophysics models. Let us stress again that perhaps the most attractive theme was the stock market. Some of the reasons for this boom are obvious, some of them are less so. We believe that one of the principal reasons is the presumed *simplicity* of the behavior of the agents. This is a crucial assumption of all the econophysics modeling and we will adhere to it, as most of the community does, too. On the other hand, it might rather be a belief than a documented fact. Economic theory likes very much trivialization of any human behavior, thus risking living in self-reinforcing isolation from reality. This fact certainly cools down our enthusiasm about agent-based modeling as a whole.

On the other hand, the very fact that there are too many existing stock-market models in the literature (we are able to mention here only a small fraction of them, see e.g. Refs. [1–7]) leads to some disappointment. It seems the state of the subject matured for some level of selection. In our view the limit order market models are good candidates for survival to the second generation of models. That is why we concentrate on some aspects of one of the models of this type.

## 2. Limit order models of market

The limit-order-driven models share several important features, which make them perhaps the candidates for “the model” of stock market behavior. Indeed, the previous models mostly suffered from arbitrariness of the prescription for the price change given the state of the market. The price thus becomes the driving dynamical quantity. On the contrary, in the limit order models it is the whole order book, which embodies the ensemble of the dynamical variables and the price is only one of the possible measurable quantities which are read off from the system.

The situation resembles e.g. the physics of surface growth, where it is the whole surface profile that must be studied, not only a handful of relevant properties, like the roughness etc. Unlike the classical attitude of thermodynamics, it is totally hopeless to find the dynamics comprising relevant observables only.

In fact, the analogy with surface growth phenomena is quite appropriate here, because the orders dropped at various positions onto the order book may be considered as particles, eventually forming complicated landscape profile described by a function  $p(x)$ , the density of orders at position  $x$ . Then, the evolution of the profile is driven by random deposition of orders (particles) and their annihilation when a buy order meets a sell order.

The second principal difference of the limit order models from other types of models consists in the absence (or at least suppression) of strategies. Actually, neither there are any agents which would be the carriers of the strategies. This may contradict the naive view of the economic activity as behavior of (atleast partially) rational agents who struggle to maximize their profit. In fact, there is no contradiction at all; the counter-intuitive conclusion comes from the excess of optimizing and complexity of the strategies pursued by humans on the trading floor. Excessively complex strategies

amount to no strategies at all, exactly in the same manner as deterministic chaos is empirically hardly distinguishable from pure stochastic randomness.

The third advantage of limit order models is the fact, that they are more rooted in obvious empirical observation. Indeed, the order book is there, the prices come from the records written in it, so it is quite natural that the true microscopic model of stock market must somehow describe the dynamics of the order book.

### 3. The Maslov model

The recent wave of interest in the limit order models started perhaps by the numerical simulations of the limit-order-driven model by Maslov [8], followed by a series of detailed empirical investigations and modeling [9–23].

In the Maslov model [8] there are two types of orders issued. First, the limit orders either to buy or to sell are recorded in the book. They are set at a specific price which does not change through the lifetime of the order. Second, market orders are issued. Sell market order is met with the currently available limit orders to buy and vice versa. The transaction is made at the price where the limit and market orders meet. The new limit orders are placed within the distance  $\Delta$  from the last price.

Numerical experiments performed by Maslov [8] found for the price changes  $\delta P$  a distribution which follows two different power laws in two well-separated regimes: for small price changes (below a certain value  $\Delta_0 \sim 4$ ), the distribution was characterized by an exponent  $\alpha_1 = -0.4 \pm 0.1$  while for large price changes ( $\delta P > \Delta_0$ ) the characteristic exponent was  $\alpha_2 = 2.0 \pm 0.2$ .

Fig. 1 and Table 1 explain the origins of the two ranges:

- In the Maslov model, the price of each new limit order was chosen within a fixed range  $\Delta \sim 4$  from the current price.
- In the runs reported in Fig. 1 and Table 1, we have used  $\Delta$  values ranging from 2 to 100 and were able to show that the break point  $\Delta_0$  (column 3 in Table 1) coincides precisely with  $\Delta$  (column 2). Therefore, we explain the presence of the  $\alpha_1 = -0.4 \pm 0.1$  in the low  $\delta P$  region as an effect of placing the limit orders in a finite range  $\Delta$  of the current price.

### 4. Reconciling Maslov measurements with analytical results

The Maslov model has been analyzed analytically by Slanina [18], assuming a continuous, constant, time-invariant limit order density. Within these assumptions, a power-law distribution with  $\alpha = 1$  over the entire prices change range was obtained. As seen above (Table 1, column 5) this is in stark contradiction with the numerical value  $\alpha_2 = 2.0 \pm 0.2$  measured by Maslov.

We suggest the following explanation of this apparent paradox: Assuming continuous limit orders distribution within the order book means practically that the price intervals between limit orders are small compared with price changes characterizing typical

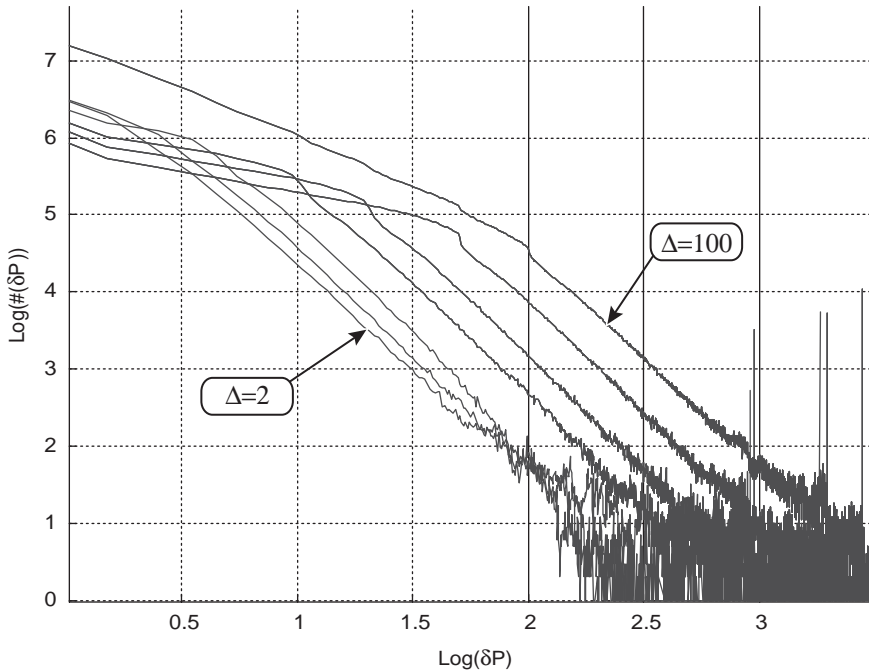


Fig. 1. Logarithmically binned histogram of price changes distribution in the Maslov model. The different curves correspond to different  $\Delta$  (the maximal distance between the current price and the new limit order prices). The point of transition (“knee”)  $\Delta_0$  between the two power-law regimes turns equal to  $\Delta$  (see also Table 1).

Table 1  
The Maslov model simulation results

$\Delta$	$\log(\Delta)$	$\log(\Delta_0)$	$\alpha_1$	$\alpha_2$
100	2.0	2.0	$-0.35 \pm 0.05$	$1.9 \pm 0.1$
50	1.70	1.7	$-0.45 \pm 0.05$	$1.90 \pm 0.05$
20	1.30	1.3	$-0.45 \pm 0.05$	$1.90 \pm 0.05$
10	1.00	1.0	$-0.45 \pm 0.05$	$1.90 \pm 0.05$
4	0.60	0.6	$-0.5 \pm 0.1$	$1.9 \pm 0.1$
2	0.30	—	—	$1.90 \pm 0.05$

The simulation was executed with several flat distributions of the limit order. The distribution ranges  $\Delta$  around the current price are listed in the first column. The distribution ranges  $\Delta$  are seen to match exactly the location  $\Delta_0$  of the transition between the two power-law regimes (columns 2 and 3 and Fig. 1). The values of  $\alpha$  for each of the regimes are listed in columns 4 and 5, respectively.

transactions. In other words, the analysis of Ref. [18] is expected to hold only when the ratio  $M$  between the typical market order volume and the limit-order volume approaches infinity:  $M \gg 1$ .

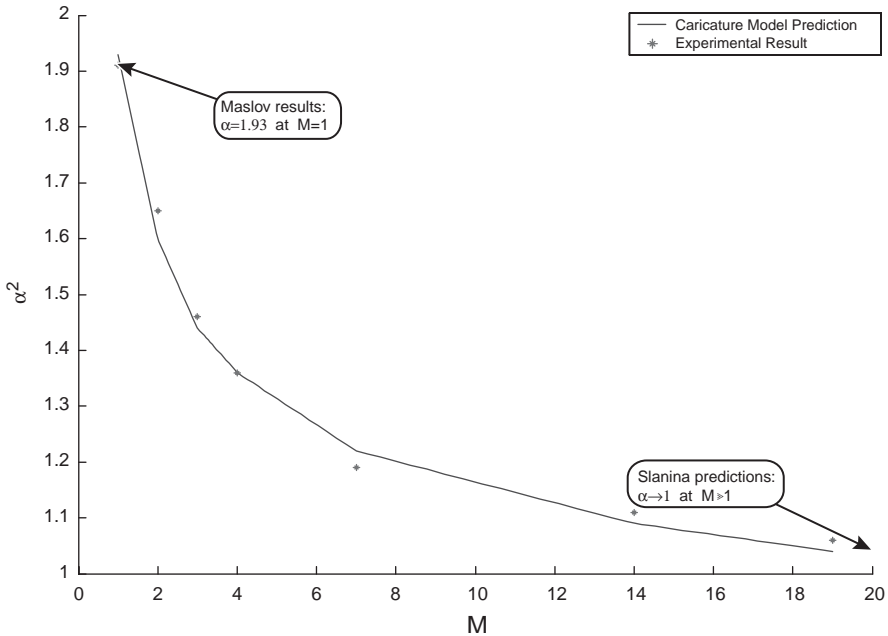


Fig. 2. The interacting gaps model theoretical predictions  $\alpha$  (continuous line) against the numerically obtained exponent  $\alpha_2$  (crosses) in the  $M$ —generalized Maslov model. Each point was obtained by averaging over 25 independent runs consisting each of 600 000 trades. The initial order book was assumed to contain one limit order at each price value.

Maslov used throughout the simulations a fixed ratio  $M = 1$  and therefore there is no wonder that the measured value for  $\alpha$  was very different ( $\alpha \approx 2$ ).

To bridge between the two extremes we suggest below a model that we call the interacting gaps model. Similar to the Maslov model ( $M = 1$ ) and the model of [18] ( $M \gg 1$ ), the interacting gaps model deals with two kinds of orders: limit and market. However the parameter  $M$  is arbitrary.

Note that the requirement that the total volume of market and limit orders are equal (a requirement common to both original Maslov model and the approach of [18]) fixes the ratio between the frequencies of the limit and market orders. More precisely, the ratio between the probability for a limit order to be generated vs. a market order to be generated is  $M$ .

We will be able to solve analytically the interacting gaps model for arbitrary  $M$ .

For  $M = 1$  the theoretical prediction  $\alpha = 1.93$  consistent with Maslov's measurements ( $\alpha_2 = 2.0 \pm 0.2$ ) while for  $M \gg 1$  the model predicts  $\alpha = 1$  consistent with analysis of Ref. [18]. For intermediate  $M$  values the model predictions are precisely confirmed by numerical simulations on a modified Maslov model with corresponding ratio  $M$  between the limit and market order frequencies (see Fig. 2).

## 5. The “interacting gaps” model and its analytic solution

The “interacting gaps” model purposes to capture the main dynamical elements of the above described models in a way that allows an analytical solution.

In order to define the interacting gaps model imagine that the order book of the Maslov model (modified to admit arbitrary  $M$ ) is represented by the real axis. Each limit order is represented on this axis by a point whose coordinate is the value of its limit price. The fundamental variables are the lengths of the intervals delimited by neighboring orders. The size distribution of these intervals corresponds to the distribution of the actual price changes.

Indeed a transaction between a market order and a limit order, changes the value of current price to a value equal to the position of that limit order. A further market order will bring the current price to the value of the next (neighboring) limit order. In general, a market order will “skip” in this way over the prices corresponding to  $M$  successive limit orders. The price changes will correspond to the size of those intervals.

The transactions will also change the actual sizes of the intervals. This will induce a dynamics governing the intervals size distribution. For instance, if one has two neighboring limit orders intervals  $n_1$  and  $n_2$ , a transaction eliminating the limit order that separates them will lead to the generation of an interval of size  $n_1 + n_2$ .

The introduction of a new limit order will substitute the limit orders “eaten” by the market order in the recent transactions by new limit orders placed in a neighborhood of size  $\Delta$  of the recent transaction. We will take  $\Delta = 1$ . Therefore, if the last transaction took place at the price value separating the  $n_1$  and  $n_2$  intervals (and consumed totally the corresponding limit order separating them) then the new intervals delimited by the placement of a new limit order will be  $n_1 - 1$  and  $n_2 + 1$  or vice-versa.

If after a market order consumed the limit order that separates 2 intervals of sizes  $n_1$  and  $n_2$  another *market* order is generated, an interval of size  $n_1 + n_2$  is generated. A subsequent limit order may modify it only by a value  $\Delta = 1$ .

The interacting gaps model disregards the topological details of the  $n_1$  and  $n_2$  intervals and considers them as arbitrary degrees of freedom governed by the following dynamics.

At each time step, 2 arbitrary intervals  $n_i$  and  $n_j$  are randomly selected and instead of them 2 other intervals are generated according to the following rules:

1. with probability  $M/(M + 1)$  the new intervals have lengths

$$n'_i = n_i - 1 \quad \text{and} \quad n'_j = n_j + 1$$

2. with probability  $1/(M + 1)$  the new intervals have length

$$n'_i = n_i + n_j - 1 \quad \text{and} \quad n'_j = 1$$

3. to avoid complications (non-positive interval lengths) we enforce the rule that if  $n_i = 1$  then

$$n'_i = n_i, \quad n'_j = n_j.$$

This system is analytically solvable by taking into account the following facts:

- the number of intervals is conserved,
- the total length of the intervals is conserved,
- the dynamics is dominated by the emergence of a very large interval  $N$  which contains most of the total length,
- the other intervals sizes are distributed by a power law  $n^{-1-\alpha}$ .

The steady distribution is then easily estimated by equating the total length that the very large interval  $N$  loses and gains during interactions with the others:

- according to rule 2, it gains a length  $n - 1$  with probability  $P(n)/(M + 1)$ ,
- according to rule 1 it loses a length 1 with probability  $M/[2(M + 1)]$ ,
- according to rule 1 (exception 3) it gains 1 with probability  $[1 - P(1)]M/[2(M + 1)]$ .

Overall, this balance implies (plugging  $P(n) \sim n^{-1-\alpha}$  [14])

$$\sum_2^{\infty} (n - 1) \frac{n^{-1-\alpha}}{M + 1} = \frac{M}{2(M + 1)}.$$

Consequently,  $\alpha$  is given by the implicit transcendental equation

$$\sum_2^{\infty} (n - 1) n^{-1-\alpha} = \frac{1}{2} M.$$

Whose solution is plotted as a function of  $M$  in Fig. 2.

One sees that this prediction fits well the numerical measurements on the modified Maslov model (the modification is that instead of having equal volumes for the market and limit orders, their ratio is  $M$ , while their relative frequency is  $1/M$ ).

## 6. Conclusions

We have studied the limit-order-driven model of Maslov [8] and its modification defined in Ref. [18]. We explained the double-power-law distribution of price changes measured by Maslov, occurring due to the finite minimum distance from current price to newly placed limit order. When this distance diminishes, the first power-law regime shrinks.

We introduced here a new schematic model, which we called interacting gaps model (IGM). The apparent contradiction with the analytic result obtained by one of us in Ref. [18] was clarified within the new model, by exposing the effect of the ratio  $M$  of the frequencies/sizes of the typical limit/market orders. Under the assumption of existence of stationary gap distribution we solved analytically the IGM in the entire range of  $M$ . The resulting power in the power-law distribution of gaps interpolates between the Maslov ( $M = 1$ ) and Slanina ( $M \gg 1$ ) results.

The predictions of this model agree with the simulations of the Maslov model generalized for entire range of  $M$  values.

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